On The Homogenized Linial Arrangement: Intersection Lattice and Genocchi Numbers

Alexander Lazar† University of Miami, Michelle L. Wachs University of Miami

Background

The Homogenized Linial Arrangement

The hyperplane arrangement

$$\mathcal{H}_{2n-3} = \{x_i - x_j = y_i \mid 1 \le i < j \le n\} \subseteq \mathbb{R}^{2n},$$

was introduced by Hetyei in 2017. Using the finite field method of Athanasiadis, Hetyei showed that its number of regions is a **median Genocchi number**.

Genocchi Numbers and Dumont Permutations

A **Dumont permutation** is a permutation σ satisfying, for all i,

$$\sigma(2i-1) \ge 2i-1, \quad \sigma(2i) < 2i.$$

The **Genocchi number** g_n is the number of Dumont permutations on $[2n-2]$.

The **median Genocchi number** h_n is the number of Dumont *derangements* on $[2n+2]$.

Our Results

Type A

We refine Hetyei's result by studying the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ and its characteristic polynomial $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$. By Zaslavsky's formula, the number of regions of $\mathcal{H}_{2n-1}$ is $|\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1)|$.

We start our study by showing that $\mathcal{L}(\mathcal{H}_{2n-1})$ is an induced subposet of the lattice of partitions of $[2n]$.

Theorem (L.-Wachs):

$$\sum_{n\ge1} \chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)x^n = \sum_{n\ge1} \frac{(t-1)_n(t-1)_{n-1}x^n}{\prod_{k=1}^n (1-k(t-k)x)},$$

where $(a)_n$ is the falling factorial $a(a-1)\cdots(a-(n-1))$.

The proof constructs a bijection from the NBC sets of $\mathcal{L}(\mathcal{H}_{2n-1})$ to a class of permutations we call D-permutations (which are discussed below), and from there to a class of excedent functions known as surjective pistols.

Plugging in $t=-1$ and $t=0$ yields the right-hand sides of the following formulas of Barsky and Dumont (1981):

$$\sum_{n\ge1} h_n x^n = \sum_{n\ge1} \frac{n!(n+1)!x^n}{\prod_{k=1}^n (1+k(k+1)x)}$$

$$\sum_{n\ge1} g_n x^n = \sum_{n\ge1} \frac{n!(n-1)!x^n}{\prod_{k=1}^n (1+k^2x)}$$

Hence, $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(0) = -g_n$ and $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1) = -h_n$.

Corollaries:

- (Hetyei) The number of regions of $\mathcal{H}_{2n-1}$ is h_n.
- (L.-Wachs) The Möbius invariant $\mu(\mathcal{L}(\mathcal{H}_{2n-1}))$ of $\mathcal{L}(\mathcal{H}_{2n-1})$ is $-g_n$.

D-Permutations

A **D-permutation** is a permutation σ satisfying, for all i,

$$\sigma(2i) \le 2i, \quad \sigma(2i-i) \ge 2i-1.$$

Theorem (L.-Wachs): The coefficient of t^{k-1} in $\chi_{\mathcal{H}_{2n-1}}(t)$ is $(-1)^k$ times the number of D-permutations on $[2n]$ with exactly k cycles.

Corollary (L.-Wachs):

- $\#\{\text{regions of } \mathcal{H}_{2n-1}\}$ is the number of D-permutations on $[2n]$.
- $\mu(\mathcal{L}(\mathcal{H}_{2n-1}))$ is -1 times the number of D-cycles on $[2n]$.

Dowling Type

Let $\omega = e^{\frac{2\pi i}{m}}$. A natural generalization of the real arrangement $\mathcal{H}_{2n-1}$ is the complex arrangement

$$\mathcal{H}^m_{2n-1} = \{x_i - \omega^\ell x_j = y_i \mid 1 \le i < j \le n, 0 \le \ell < m\} \cup \{x_i = y_i \mid 1 \le i \le n\},$$

which we call the **homogenized Linial-Dowling arrangement**.

We show that $\mathcal{L}(\mathcal{H}^m_{2n-1})$ is an induced subposet of the Dowling lattice $Q_{2n-1}(\mathbb{Z}/m\mathbb{Z})$.

Theorem (L.-Wachs):

$$\sum_{n\ge1} \chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t)x^n = \sum_{n\ge1} \frac{(t-1)_{n,m}(t-m)_{n-1,m}x^n}{\prod_{k=1}^n (1-mk(t-mk)x)},$$

where $(a)_{n,m} = a(a-m)\cdots(a-(n-1)m)$.

When $m=1$, $\mathcal{L}(\mathcal{H}^1_{2n-1}) \cong \mathcal{L}(\mathcal{H}_{2n-1})$.

When $m=2$, $\mathcal{H}^2_{2n-1}$ is the complexification of the **type B homogenized Linial arrangement**

$$\mathcal{H}^B_{2n-1} = \{x_i \pm x_j = y_i \mid 1 \le i < j \le n\} \cup \{x_i = y_i \mid 1 \le i \le n\} \subseteq \mathbb{R}^{2n}.$$

By Zaslavsky's formula, setting $m=2$ and $t=-1$ in the Theorem gives the following enumerative result.

Corollary (L.-Wachs): Let r^B_n be the number of regions of $\mathcal{H}^B_{2n-1}$. Then

$$\sum_{n\ge1} r^B_n t^n = \sum_{n\ge1} \frac{(2n)!x^n}{\prod_{k=1}^n (1+2k(2k+1)x)}.$$

Gandhi Polynomials

The Gandhi polynomials $G_n(x)$ are the recursively-defined polynomials given by $G_1(x) = x^2$ and $G_n(x) = x^2(G_{n-1}(x+1) - G_{n-1}(x))$. They were shown to satisfy $G_n(1) = g_n$ by Carlitz (1972) and Riordan and Stein (1973).

Theorem (L.-Wachs): $\mu(\mathcal{L}(\mathcal{H}^m_{2n-1})) = -m^{2n-1}G_n(m^{-1})$.

Decorated D-Permutations

An m**-labeled D-permutation** is a D-permutation with certain entries given decorations from the set $\{0,\ldots,m-1\}$.

Theorem (L.-Wachs): The coefficient of t^{k-1} in $\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t)$ is $(-1)^k$ times the number of m-labeled D-permutations on $[2n]$ with exactly k cycles.

http://www.math.miami.edu/~alazar

alazar@math.miami.edu